

## Strong commutativity preserving derivations on Lie ideals of prime $\Gamma$ -rings

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ABSTRACT. Let  $M$  be a  $\Gamma$ -ring and  $S \subseteq M$ . A mapping  $f : M \rightarrow M$  is called *strong commutativity preserving* on  $S$  if  $[f(x), f(y)]_\alpha = [x, y]_\alpha$ , for all  $x, y \in S$ ,  $\alpha \in \Gamma$ . In the present paper, we investigate the commutativity of the prime  $\Gamma$ -ring  $M$  of characteristic not 2 with center  $Z(M) \neq (0)$  admitting a derivation which is strong commutativity preserving on a nonzero square closed Lie ideal  $U$  of  $M$ . Moreover, we also obtain a related result when a mapping  $d$  is assumed to be a derivation on  $U$  satisfying the condition  $d(u) \circ_\alpha d(v) = u \circ_\alpha v$ , for all  $u, v \in U$ ,  $\alpha \in \Gamma$ .

### 1. INTRODUCTION

Nobusawa [13] developed the concept of a gamma ring and then Barnes [1] weakened slightly the defining conditions for a gamma ring. After these definitions a number of mathematicians have studied on gamma rings in the sense of Barnes and Nobusawa and get results parallel to the ring theory (see for example [1], [11], [9]).

Let  $R$  be any ring. The symbol  $[a, b]$  denotes  $ab - ba$  for  $a, b \in R$ .  $R$  is called *prime* if  $aRb = (0)$  implies either  $a = 0$  or  $b = 0$ , and  $R$  is called *semiprime* if  $aRa = (0)$  implies  $a = 0$ . An additive mapping  $d$  is called a *derivation* on  $R$  if

$$d(ab) = d(a)b + ad(b)$$

holds for all  $a, b \in R$ .

A mapping  $f$  is said to be *commutativity preserving* on  $R$  if  $[f(a), f(b)] = 0$  whenever  $[a, b] = 0$ , for all  $a, b \in R$ . In 1976, Watkins [14] obtained the first result on commutativity preserving maps for a  $n \times n$  matrix algebra when  $n \geq 4$  and  $f$  is a monomorphism on  $R$ . Recently, the study of commutativity preserving maps has become an active research area in ring theory (see for example [4], [6], [8], [12] and references therein).

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Let  $S$  be a subset of  $R$ . A map  $f$  is called *strong commutativity preserving* (SCP) on  $S$  if  $[f(a), f(b)] = [a, b]$ , for all  $a, b \in S$ . Clearly, a map that is strong commutativity preserving on a set  $S$  is also commutativity preserving on  $S$ , but the inverse is not true in general. The notion of a strong commutativity preserving map was first introduced by H.E. Bell and G. Mason [3]. Later, H.E. Bell and M.N. Daif [2] proved that if a semiprime ring  $R$  admits a nonzero derivation which is strong commutativity preserving on a right ideal  $\rho$  of  $R$ , then  $\rho \subseteq Z(R)$  where  $Z(R)$  is the center of  $R$ . In particular,  $R$  is commutative if  $\rho = R$ . M. Brešar and C.R. Miers [5] characterized SCP additive maps on a semiprime ring. In [10], Brešar and Miers's result was extended to Lie ideals of prime rings by J.-S. Lin and C.-K. Liu. Later, Q. Deng and M. Ashraf [7] proved that if there exists a derivation  $d$  of a semiprime ring  $R$  and a mapping  $f : I \rightarrow R$  defined on a nonzero ideal  $I$  of  $R$  such that  $[f(a), d(b)] = [a, b]$ , for all  $a, b \in I$ , then  $R$  contains a nonzero central ideal. They also showed that  $R$  is commutative when  $I = R$ . There are lots of generalizations similar to these results can be found in the literature.

Recently, X. Xu, J. Ma and Y. Zhou [15] proved that a semiprime  $\Gamma$ -ring with a strong commutativity preserving derivation on itself must be commutative and that a strong commutativity preserving endomorphism  $\sigma$  on a semiprime  $\Gamma$ -ring  $M$  must have the form  $\sigma(a) = a + \xi(a)$  ( $a \in M$ ) where  $\xi$  is a map from  $M$  into its center, which extends some results by Bell and Daif to semiprime  $\Gamma$ -rings.

Motivated by all these results, in the present paper, we study strong commutativity preserving derivations on a nonzero square closed Lie ideal of prime  $\Gamma$ -rings and prove that if  $M$  is a prime  $\Gamma$ -ring of characteristic not 2 such that its center  $Z(M) \neq (0)$  and  $d$  is a SCP derivation on a nonzero square closed Lie ideal  $U$  of  $M$ , then  $U \subseteq Z(M)$ . In particular,  $M$  is commutative if  $U = M$ . Moreover, we also obtain the same result when a mapping  $d$  is assumed to be a derivation on  $U$  satisfying the condition  $d(u) \circ_\alpha d(v) = u \circ_\alpha v$ , for all  $u, v \in U$ ,  $\alpha \in \Gamma$ .

## 2. PRELIMINARIES

Before giving our results, we first present some preliminary definitions. In this paper,  $M$  will represent a  $\Gamma$ -ring in the sense of Barnes [1] unless otherwise stated.

An additive subgroup  $K$  of a  $\Gamma$ -ring  $M$  is called a *left (resp. right) ideal* of  $M$  if  $M\Gamma K \subseteq K$  (resp.  $K\Gamma M \subseteq K$ ). A left ideal  $K$  of a  $\Gamma$ -ring  $M$  is called an *ideal* of  $M$  if it is also a right ideal of  $M$ . The set of all elements  $a$  satisfying  $a\alpha b = b\alpha a$  for all  $b \in M$  and  $\alpha \in \Gamma$  is called the *center* of  $M$ .

A  $\Gamma$ -ring  $M$  is said to be *prime* if  $a\Gamma M\Gamma b = (0)$  for  $a, b \in M$  implies that  $a = 0$  or  $b = 0$ . An additive mapping  $d$  is called a *derivation* on  $M$  if  $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ , for all  $a, b \in M$  and  $\alpha \in \Gamma$ .

Let  $M$  be a  $\Gamma$ -ring and  $a, b \in M$ ,  $\alpha \in \Gamma$ . The commutator of  $a$  and  $b$  with respect to  $\alpha$  is defined as the element  $a\alpha b - b\alpha a$  and denoted by  $[a, b]_\alpha$ . According to this definition we have the following equations,

$$(1) \quad [a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha [b, c]_\beta + a\alpha c\beta b - a\beta c\alpha b,$$

$$(2) \quad [a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha [a, c]_\beta + b\beta a\alpha c - b\alpha a\beta c,$$

where  $a, b, c \in M$ ,  $\alpha, \beta \in \Gamma$ . Similarly, the anti-commutator of  $a$  and  $b$  with respect to  $\alpha$  is defined as the element  $a\alpha b + b\alpha a$  and denoted by  $a \circ_\alpha b$ . According to this definition we have the following equations,

$$\begin{aligned} (a\alpha b) \circ_\beta c &= a\alpha (b \circ_\beta c) - [a, c]_\beta \alpha b + a\alpha c\beta b - a\beta c\alpha b \\ &= (a \circ_\beta c)\alpha b + a\alpha [b, c]_\beta + a\beta c\alpha b - a\alpha c\beta b, \end{aligned}$$

$$\begin{aligned} a \circ_\beta (b\alpha c) &= (a \circ_\beta b)\alpha c - b\alpha [a, c]_\beta + b\beta a\alpha c - b\alpha a\beta c \\ &= b\alpha (a \circ_\beta c) + [a, b]_\beta \alpha c + b\alpha a\beta c - b\beta a\alpha c, \end{aligned}$$

where  $a, b, c \in M$ ,  $\alpha, \beta \in \Gamma$ .

An additive subgroup  $U$  of a  $\Gamma$ -ring  $M$  is called a *Lie ideal* if  $[u, m]_\alpha \in U$ , for all  $u \in U$ ,  $m \in M$  and  $\alpha \in \Gamma$ . A Lie ideal  $U$  of  $M$  is said to be a *square closed Lie ideal* of  $M$ , if  $u\alpha u \in U$  for all  $u \in U$  and  $\alpha \in \Gamma$ . Clearly,  $u\alpha v + v\alpha u \in U$ , for all  $u, v \in U$ ,  $\alpha \in \Gamma$ . Similarly, we have  $u\alpha v - v\alpha u \in U$ . Moreover, by using these relations, we get  $2u\alpha v \in U$  which will be used in the whole paper frequently.

A map  $f$  from a  $\Gamma$ -ring  $M$  into itself is called *strong commutativity preserving* (SCP) on a subset  $S$  of  $M$  if  $[f(a), f(b)]_\alpha = [a, b]_\alpha$  holds for all  $a, b \in S$  and  $\alpha \in \Gamma$ .

### 3. THE RESULTS

First, we work on SCP derivations on Lie ideals of prime  $\Gamma$ -rings. The following lemma will play an crucial role in the proofs of our main theorems.

**Lemma 3.1.** *Let  $M$  be a prime  $\Gamma$ -ring and  $Z(M) \neq (0)$ . Then the equations*

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha [b, c]_\beta,$$

$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha [a, c]_\beta$$

hold for all  $a, b, c \in M$ ,  $\alpha, \beta \in \Gamma$ .

*Proof.* For any  $c \in M$ ,  $\alpha, \beta \in \Gamma$ , the symbol  $[\alpha, \beta]_c$  denotes  $\alpha c \beta - \beta c \alpha$ . Then, the commutator formulas in (1) and (2) become

$$(3) \quad [a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha [b, c]_\beta + a[\alpha, \beta]_c b$$

and

$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha [a, c]_\beta + b[\beta, \alpha]_a c,$$

for all  $a, b, c \in M$ ,  $\alpha, \beta \in \Gamma$ .

Since  $Z(M) \neq (0)$ , there exists a nonzero element  $x$  in  $Z(M)$ . Thus,

$$\begin{aligned} x\gamma y\delta a\alpha c\beta b &= y\gamma x\delta a\alpha c\beta b = y\gamma a\delta x\alpha c\beta b \\ &= y\gamma a\delta c\alpha x\beta b = y\gamma a\delta c\alpha b\beta x \\ &= y\gamma a\delta x\beta c\alpha b = y\gamma x\delta a\beta c\alpha b \\ &= x\gamma y\delta a\beta c\alpha b, \end{aligned}$$

for all  $a, b, c, y \in M$ ,  $\alpha, \beta, \gamma, \delta \in \Gamma$ . Then we have that

$$(4) \quad x\gamma y\delta a[\alpha, \beta]_c b = 0,$$

for all  $a, b, c, y \in M$ ,  $\alpha, \beta, \gamma, \delta \in \Gamma$ . Multiplying the two sides of (3) by  $x\gamma y\delta$  from the left hand side, and then comparing with (4) we get for all  $a, b, c, y \in M$ ,  $\alpha, \beta, \gamma, \delta \in \Gamma$

$$x\gamma y\delta[a\alpha b, c]_\beta = x\gamma y\delta[a, c]_\beta \alpha b + x\gamma y\delta a\alpha[b, c]_\beta.$$

That is  $x\Gamma M\Gamma([a\alpha b, c]_\beta - [a, c]_\beta \alpha b - a\alpha[b, c]_\beta) = 0$ , for all  $a, b, c \in M$ ,  $\alpha, \beta \in \Gamma$ . Since  $M$  is prime and  $x$  is nonzero, we have

$$[a\alpha b, c]_\beta - [a, c]_\beta \alpha b - a\alpha[b, c]_\beta = 0,$$

for all  $a, b, c \in M$ ,  $\alpha, \beta \in \Gamma$ . For the second equation, one can use the same method above, and this completes the proof.  $\square$

Now, we can give a similar result for the anti-commutator formulas of  $\Gamma$ -rings.

**Lemma 3.2.** *Let  $M$  be a prime  $\Gamma$ -ring in the sense of Barnes and  $Z(M) \neq (0)$ . Then the equations*

$$\begin{aligned} (a\alpha b) \circ_\beta c &= a\alpha(b \circ_\beta c) - [a, c]_\beta \alpha b \\ &= (a \circ_\beta c)\alpha b + a\alpha[b, c]_\beta, \\ a \circ_\beta (b\alpha c) &= (a \circ_\beta b)\alpha c - b\alpha[a, c]_\beta \\ &= b\alpha(a \circ_\beta c) + [a, b]_\beta \alpha c \end{aligned}$$

hold for all  $a, b, c \in M$ ,  $\alpha, \beta \in \Gamma$ .

*Proof.* It can be proved by using the techniques of Lemma 3.1.  $\square$

We need the following results to prove our main theorems.

**Lemma 3.3.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 with the center  $Z(M) \neq (0)$  and  $U$  be a Lie ideal of  $M$ . If  $U \not\subseteq Z(M)$ , then there exists an ideal  $K$  of  $M$  such that  $[K, M]_\Gamma \subseteq U$  but  $[K, M]_\Gamma \not\subseteq Z(M)$ .*

*Proof.* First, we show that the Lie product of  $U$  by itself is different from zero. Suppose that  $[U, U]_\Gamma = (0)$ . Then we have  $[a, [a, m]_\alpha]_\beta = 0$ , for all  $a \in U$ ,  $m \in M$  and  $\alpha, \beta \in \Gamma$ . Replacing  $m$  by  $m\gamma x$  for  $\gamma \in \Gamma$  and  $x \in M$ , we get

$$(5) \quad [a, m]_\beta \gamma [a, x]_\alpha + [a, m]_\alpha \gamma [a, x]_\beta = 0.$$

Now, replacing  $\beta$  by  $\alpha$  in (5) we have  $[a, m]_\alpha \gamma [a, x]_\alpha = 0$ , for all  $a \in U$ ,  $m, x \in M$  and  $\alpha, \gamma \in \Gamma$ . Replacing  $x$  by  $y\delta x$  for  $y \in M$  and  $\delta \in \Gamma$  in the last equation, we get  $[a, m]_\alpha \Gamma M \Gamma [a, x]_\alpha = (0)$ , for all  $a \in U$ ,  $m, x \in M$  and  $\alpha \in \Gamma$ . Therefore, we have  $U \subseteq Z(M)$  since  $M$  is prime. But this contradicts with the hypothesis of the theorem. Hence, there exist  $u, v \in U$  and  $\beta \in \Gamma$  such that  $[u, v]_\beta \neq 0$ .

Let  $K := M\Gamma[u, v]_\beta \Gamma M$  and  $T(U) := \{x \in M \mid [x, M]_\Gamma \subseteq U\}$ . Then, it is clear that  $K \neq (0)$  is an ideal of  $M$ ;  $T(U)$  is a Lie ideal and a subring of  $M$ . Moreover,  $U \subseteq T(U)$ . Since  $[u, v\gamma m]_\beta = [u, v]_\beta \gamma m + v\gamma [u, m]_\beta$  for all  $m \in M$  and  $\gamma \in \Gamma$ , we get  $[u, v]_\beta \Gamma M \subseteq T(U)$ . Hence,

$$\left[ [u, v]_\beta \alpha m, n \right]_\gamma \in T(U),$$

for all  $n, m \in M$  and  $\alpha, \gamma \in \Gamma$ . Expanding this we get  $n\gamma [u, v]_\beta \alpha m \in T(U)$  for all  $n, m \in M$  and  $\alpha, \gamma \in \Gamma$ . Then, we have  $M\Gamma[u, v]_\beta \Gamma M = K \subseteq T(U)$  which yields to  $[K, M]_\Gamma \subseteq U$ .

Now, suppose  $[K, M]_\Gamma \subseteq Z(M)$ . Therefore, we have  $[K, [K, M]_\Gamma]_\Gamma = (0)$  and using the same argument above we get  $K \subseteq Z(M)$ . Let  $x \in M$ . Then  $n\alpha k\gamma m \in K$  for all  $n, m \in M$ ,  $k \in K$  and  $\alpha, \gamma \in \Gamma$ . Since  $K \subseteq Z(M)$  we have  $[x, n\alpha k\gamma m]_\delta = 0$ . Expanding this we get  $K\Gamma M\Gamma[x, M]_\Gamma = (0)$ . Therefore,  $x \in Z(M)$  since  $M$  is prime and  $K \neq (0)$ . But this contradicts with  $U \not\subseteq Z(M)$ . This completes the proof.  $\square$

**Lemma 3.4.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 with the center  $Z(M) \neq (0)$  and  $U$  be a Lie ideal of  $M$ . If  $U \not\subseteq Z(M)$  and  $a, b \in M$  such that  $a\Gamma U\Gamma b = (0)$ , then either  $a = 0$  or  $b = 0$ .*

*Proof.* By Lemma 3.3, there exists an ideal  $K$  of  $M$  such that  $[K, M]_\Gamma \subseteq U$  but  $[K, M]_\Gamma \not\subseteq Z(M)$ . Let  $u \in U$ ,  $k \in K$ ,  $m \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ . Then, we have

$$[k\alpha\beta u, m]_\gamma \in [K, M]_\Gamma \subseteq U.$$

It follows from that

$$\begin{aligned} 0 &= a\lambda[k\alpha\beta u, m]_\gamma \epsilon b = a\lambda k\alpha\beta [u, m]_\gamma \epsilon b + a\lambda[k\alpha\alpha, m]_\gamma \beta u \epsilon b \\ &= a\lambda k\alpha\gamma m \beta u \epsilon b - a\lambda m \gamma k\alpha\alpha \beta u \epsilon b \\ &= a\lambda k\alpha\gamma m \beta u \epsilon b, \end{aligned}$$

for all  $u \in U$ ,  $k \in K$ ,  $m \in M$  and  $\alpha, \beta, \gamma, \lambda, \epsilon \in \Gamma$ . Therefore, we get  $a\Gamma K\Gamma a = (0)$  or  $U\Gamma b = (0)$  since  $M$  is prime. In the first case, we see that  $a$  must be zero by using the primeness of  $M$ . In the second case, we get

$$[u, m]_\alpha \gamma b = 0,$$

for all  $u \in U$ ,  $m \in M$  and  $\alpha, \gamma \in \Gamma$ . Expanding this we have

$$[u\gamma b, m]_\alpha - u\gamma [b, m]_\alpha = 0,$$

that is  $u\gamma m\alpha b = 0$ , for all  $u \in U$ ,  $m \in M$  and  $\alpha, \gamma \in \Gamma$ . Therefore,  $b = 0$  since  $M$  is prime and  $U \neq (0)$ .  $\square$

**Lemma 3.5.** *Let  $M$  be a prime  $\Gamma$ -ring with the center  $Z(M) \neq (0)$  and  $x \in M$ . If  $a \in Z(M)$  and  $a\gamma x \in Z(M)$  for all  $\gamma \in \Gamma$ , then  $a = 0$  or  $x \in Z(M)$ .*

*Proof.* Suppose that  $a \neq 0$ . Since  $a\gamma x \in Z(M)$ , we have  $[a\gamma x, m]_\delta = 0$  for all  $m \in M$  and  $\delta, \gamma \in \Gamma$ . Expanding this we get  $a\gamma[x, m]_\delta = 0$ . Replacing  $m$  by  $m\beta n$  for  $n \in M$  and  $\beta \in \Gamma$  we conclude that  $x \in Z(M)$  since  $M$  is prime. This completes the proof.  $\square$

**Lemma 3.6.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 with the center  $Z(M) \neq (0)$  and  $U$  be a Lie ideal of  $M$ . If  $[U, U]_\Gamma \subseteq Z(M)$ , then  $U \subseteq Z(M)$ .*

*Proof.* By hypothesis we have  $[u, [u, x]_\alpha]_\beta \in Z(M)$  for all  $u \in U$ ,  $x \in M$  and  $\alpha, \beta \in \Gamma$ . Since

$$[u, [u, x]_\alpha]_\beta \gamma u = [u, [u, x]_\alpha \gamma u]_\beta = [u, [u, x\gamma u]_\alpha]_\beta$$

and  $[u, [u, x\gamma u]_\alpha]_\beta \in [U, U]_\Gamma$ , we have  $[u, [u, x]_\alpha]_\beta \gamma u \in Z(M)$ . Therefore, we get  $[u, [u, x]_\alpha]_\beta = 0$  or  $u \in Z(M)$  by Lemma 3.5. Now, let  $[u, [u, x]_\alpha]_\beta = 0$  for all  $x \in M$ ,  $\alpha, \beta \in \Gamma$  and for some  $u \in U$ . Replacing  $x$  by  $x\gamma m$  we get

$$(6) \quad [u, x]_\beta \gamma [u, m]_\alpha + [u, x]_\alpha \gamma [u, m]_\beta = 0,$$

for all  $x, m \in M$  and  $\alpha, \beta, \gamma \in \Gamma$ . Replacing  $\beta$  by  $\alpha$  in the equation (6) we get  $[u, x]_\alpha \gamma [u, m]_\alpha = 0$  since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Replacing  $m$  by  $m\delta n$  for  $n \in M$ ,  $\delta \in \Gamma$  in the last equation, we conclude that  $u \in Z(M)$  since  $M$  is prime. Consequently, we see that  $U$  must be a subset of  $Z(M)$ .  $\square$

**Theorem 3.1.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 and  $U$  be a nonzero square closed Lie ideal of  $M$ . If  $d$  is a SCP derivation on  $U$ , then  $U \subseteq Z(M)$  or  $Z(M) = (0)$ .*

*Proof.* Suppose that  $Z(M) \neq (0)$ . We have  $[d(x), d(y)]_\alpha = [x, y]_\alpha$  for all  $x, y \in U$  and  $\alpha \in \Gamma$  by hypothesis. Replacing  $y$  by  $2y\beta z$  for  $z \in U$  and  $\beta \in \Gamma$ , we get

$$[d(x), d(2y\beta z)]_\alpha = [x, 2y\beta z]_\alpha,$$

for all  $x, y, z \in U$  and  $\alpha, \beta \in \Gamma$ . By applying Lemma 3.1, we expand the last equation and we get

$$(7) \quad d(y)\beta[d(x), z]_\alpha + [d(x), y]_\alpha \beta d(z) = 0,$$

since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Replacing  $z$  by  $2z\gamma t$  for  $z, t \in U$  and  $\gamma \in \Gamma$  in the equation (7) we obtain that

$$(8) \quad \begin{aligned} d(y)\beta[d(x), z]_\alpha \gamma t + d(y)\beta z \gamma [d(x), t]_\alpha + [d(x), y]_\alpha \beta d(z) \gamma t \\ + [d(x), y]_\alpha \beta z \gamma d(t) = 0, \end{aligned}$$

since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Multiplying the two sides of (7) by  $\gamma t$  from the right hand side, we have

$$(9) \quad d(y)\beta[d(x), z]_\alpha\gamma t + [d(x), y]_\alpha\beta d(z)\gamma t = 0,$$

for all  $x, y, z, t \in U$  and  $\alpha, \beta, \gamma \in \Gamma$ . Comparing (9) with (8), we have that

$$d(y)\beta z\gamma[d(x), t]_\alpha + [d(x), y]_\alpha\beta z\gamma d(t) = 0,$$

for all  $x, y, z, t \in U$  and  $\alpha, \beta, \gamma \in \Gamma$ . Since  $U$  is a nonzero square closed Lie ideal of  $M$ , we have  $[U, U]_\Gamma$  is a nonzero square closed Lie ideal of  $M$ , too. Writing  $t = d(x)$  for  $x \in [U, U]_\Gamma$ , we obtain that

$$(10) \quad [d(x), y]_\alpha\beta z\gamma d^2(x) = 0,$$

for all  $y, z \in U$ ,  $x \in [U, U]_\Gamma$  and  $\alpha, \beta, \gamma \in \Gamma$ . If we replace  $y$  by  $d(y)$  for  $y \in [U, U]_\Gamma$  in the equation (10), we obtain  $[x, y]_\alpha\Gamma U\Gamma d^2(x) = (0)$  for all  $x, y \in [U, U]_\Gamma$ , and  $\alpha \in \Gamma$  since  $d$  is SCP on  $U$ . Therefore,

$$[x, y]_\alpha\beta 2[m, z]_\alpha\Gamma U\Gamma[x, y]_\alpha\beta 2[m, z]_\alpha = (0),$$

since

$$[x, y]_\alpha\Gamma U\Gamma[d^2(x), d^2(y)]_\alpha\beta 2[m, z]_\alpha = (0),$$

for all  $x, y \in [U, U]_\Gamma$ ,  $m \in M$ ,  $z \in U$  and  $\alpha, \beta \in \Gamma$ . Since  $M$  is a  $\Gamma$ -ring of characteristic not 2, we have  $[x, y]_\alpha\beta[m, z]_\alpha = 0$  by Lemma 3.4. Replacing  $m$  by  $m\gamma t$  for  $t \in M$  and  $\gamma \in \Gamma$  we get

$$[x, y]_\alpha\beta m\gamma[t, z]_\alpha = 0,$$

for all  $x, y \in [U, U]_\Gamma$ ,  $m, t \in M$ ,  $z \in U$  and  $\alpha, \beta, \gamma \in \Gamma$ . By the primeness of the  $\Gamma$ -ring  $M$ , we get either  $[x, y]_\alpha = 0$  or  $[t, z]_\alpha = 0$ , for all  $x, y \in [U, U]_\Gamma$ ,  $z \in U$ ,  $t \in M$  and  $\alpha \in \Gamma$ . In the second case, we see that  $z \in Z(M)$  that is  $U \subseteq Z(M)$ . In the first case, using Lemma 3.6, we have  $[U, U]_\Gamma \subseteq Z(M)$ . Consequently, applying Lemma 3.6 again, we get that  $U \subseteq Z(M)$  which completes the proof.  $\square$

In particular, if we take  $U = M$ , then Theorem 3.1 gives a commutativity criterion as follows.

**Corollary 3.1.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 and  $d$  be a derivation of  $M$ . If  $Z(M) \neq (0)$  and  $d$  is SCP on  $M$ , then  $M$  is commutative.*

Since we can use the similar techniques of Theorem 3.1, we can obtain the following theorems which partially generalize the result of Bell and Daif to prime  $\Gamma$ -rings.

**Theorem 3.2.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 and  $U$  be a nonzero square closed Lie ideal of  $M$ . If  $[d(x), d(y)]_\alpha = -[x, y]_\alpha$  for all  $x, y \in U$  and  $\alpha \in \Gamma$ , then  $U \subseteq Z(M)$  or  $Z(M) = (0)$ .*

*Proof.* It can be proved easily by using the same method in Theorem 3.1.  $\square$

**Corollary 3.2.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 and  $d$  be a derivation of  $M$ . If  $Z(M) \neq (0)$  and  $[d(x), d(y)]_\alpha = -[x, y]_\alpha$  for all  $x, y \in M$ ,  $\alpha \in \Gamma$ , then  $M$  is commutative.*

**Theorem 3.3.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 and  $U$  be a nonzero square closed Lie ideal of  $M$ . If  $d$  is a derivation of  $M$  such that  $d(x) \circ_\alpha d(y) = x \circ_\alpha y$  for all  $x, y \in U$  and  $\alpha \in \Gamma$ , then  $U \subseteq Z(M)$  or  $Z(M) = (0)$ .*

*Proof.* Suppose that  $Z(M) \neq (0)$ . By the hypothesis we obtain that

$$(11) \quad d(x) \circ_\alpha d(y) - x \circ_\alpha y = 0,$$

for all  $x, y \in U$  and  $\alpha \in \Gamma$ . Replacing  $x$  by  $2x\beta z$  for  $z \in U$ ,  $\beta \in \Gamma$  in the equation (11) we get

$$(12) \quad d(x)\beta[z, d(y)]_\alpha - [x, d(y)]_\alpha\beta d(z) + 2x\beta y\alpha z = 0,$$

since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Taking  $2z\gamma x$  for  $z$  in the equation (12) we have

$$(13) \quad d(x)\beta[z, d(y)]_\alpha\gamma x + d(x)\beta z\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha\beta d(z)\gamma x \\ - [x, d(y)]_\alpha\beta z\gamma d(x) + 2x\beta y\alpha z\gamma x = 0,$$

for all  $x, y, z \in U$ ,  $\alpha, \beta, \gamma \in \Gamma$ . Multiplying the two sides of (12) by  $\gamma x$  from the right hand side, we get

$$(14) \quad d(x)\beta[z, d(y)]_\alpha\gamma x - [x, d(y)]_\alpha\beta d(z)\gamma x + 2x\beta y\alpha z\gamma x = 0,$$

for all  $x, y, z \in U$  and  $\alpha, \beta, \gamma \in \Gamma$ . If we compare (13) and (14), we have that

$$(15) \quad d(x)\beta z\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha\beta z\gamma d(x) = 0,$$

for all  $x, y, z \in U$  and  $\alpha, \beta, \gamma \in \Gamma$ . Replacing  $z$  by  $2z\sigma[x, d(y)]_\alpha$  for  $y \in [U, U]_\Gamma$  and  $\sigma \in \Gamma$  in the equation (15) we get

$$(16) \quad d(x)\beta z\sigma[x, d(y)]_\alpha\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha\beta z\sigma[x, d(y)]_\alpha\gamma d(x) = 0,$$

since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Taking  $\sigma$  for  $\gamma$  in (15) we have

$$(17) \quad d(x)\beta z\sigma[x, d(y)]_\alpha = [x, d(y)]_\alpha\beta z\sigma d(x).$$

If we use the equation (17) in the equation (16) we get

$$[x, d(y)]_\alpha\beta z\sigma d(x)\gamma[x, d(y)]_\alpha = [x, d(y)]_\alpha\beta z\sigma[x, d(y)]_\alpha\gamma d(x)$$

and so

$$(18) \quad [x, d(y)]_\alpha\beta z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0,$$

for all  $x, z \in U$ ,  $y \in [U, U]_\Gamma$  and  $\alpha, \beta, \gamma, \sigma \in \Gamma$ . Taking  $\beta = \gamma$  in (18), we get

$$(19) \quad [x, d(y)]_\alpha\gamma z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0,$$



for all  $x, z \in U$ ,  $y \in [U, U]_\Gamma$  and  $\alpha, \gamma, \sigma \in \Gamma$ . Multiplying the equation (19) on the left by  $d(x)\gamma$  for  $x \in [U, U]_\Gamma$ , we have

$$(20) \quad d(x)\gamma[x, d(y)]_\alpha\gamma z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0.$$

Taking  $2d(x)\gamma z$  for  $z$  in (19) we obtain that

$$(21) \quad [x, d(y)]_\alpha\gamma d(x)\gamma z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0,$$

for all  $z \in U$ ,  $x, y \in [U, U]_\Gamma$  and  $\alpha, \gamma, \sigma \in \Gamma$  since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Subtracting (21) from (20) we see that

$$[d(x), [x, d(y)]_\alpha]_\gamma\gamma z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0,$$

for all  $z \in U$ ,  $x, y \in [U, U]_\Gamma$  and  $\alpha, \gamma, \sigma \in \Gamma$ . Therefore, by Lemma 3.4 we have that

$$(22) \quad [d(x), [x, d(y)]_\alpha]_\gamma = 0,$$

for all  $x, y \in [U, U]_\Gamma$  and  $\alpha, \gamma \in \Gamma$ . Replacing  $z$  by  $x$  for  $x \in [U, U]_\Gamma$  and  $\beta = \gamma$  in (12) and using the equation (22) we conclude that  $x\Gamma[U, U]_\Gamma\Gamma x = (0)$  for all  $x \in [U, U]_\Gamma$  since  $M$  is a  $\Gamma$ -ring of characteristic not 2. We know that  $[U, U]_\Gamma$  is a nonzero square closed Lie ideal of  $M$ . So by using Lemma 3.4 we get either  $x = 0$  for all  $x \in [U, U]_\Gamma$  or  $[U, U]_\Gamma \subseteq Z(M)$ . The first case contradicts with the hypothesis  $[U, U]_\Gamma \neq (0)$ . Then we have that  $[U, U]_\Gamma \subseteq Z(M)$ . Hence, applying Lemma 3.6 we obtain that  $U \subseteq Z(M)$ . This completes the proof.  $\square$

**Corollary 3.3.** *Let  $d$  be a derivation of a prime  $\Gamma$ -ring  $M$  of characteristic not 2. If  $d(x) \circ_\alpha d(y) = x \circ_\alpha y$  for all  $x, y \in M$ ,  $\alpha \in \Gamma$  and  $Z(M) \neq (0)$ , then  $M$  is commutative.*

**Theorem 3.4.** *Let  $M$  be a prime  $\Gamma$ -ring of characteristic not 2 and  $U$  be a nonzero square closed Lie ideal of  $M$ . If  $d$  is a derivation of  $M$  such that  $d(x) \circ_\alpha d(y) = -(x \circ_\alpha y)$  for all  $x, y \in U$ ,  $\alpha \in \Gamma$ , then  $U \subseteq Z(M)$  or  $Z(M) = (0)$ .*

*Proof.* Suppose that  $Z(M) \neq (0)$ . By the hypothesis we have that

$$(23) \quad d(x) \circ_\alpha d(y) + x \circ_\alpha y = 0,$$

for all  $x, y \in U$  and  $\alpha \in \Gamma$ . Replacing  $x$  by  $2x\beta z$  for  $z \in U$ ,  $\beta \in \Gamma$  in the equation (23) we get

$$(24) \quad d(x)\beta[z, d(y)]_\alpha - [x, d(y)]_\alpha\beta d(z) + 2x\beta z\alpha y = 0,$$

since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Taking  $2z\gamma x$  for  $x$  in the equation (24) we have

$$(25) \quad \begin{aligned} d(z)\gamma x\beta[z, d(y)]_\alpha + z\gamma d(x)\beta[z, d(y)]_\alpha - [z, d(y)]_\alpha\gamma x\beta d(z) \\ - z\gamma[x, d(y)]_\alpha\beta d(z) + 2z\gamma x\beta z\alpha y = 0. \end{aligned}$$

Multiplying the two sides of (24) by  $z\gamma$  from the left hand side, we get

$$(26) \quad z\gamma d(x)\beta[z, d(y)]_\alpha - z\gamma[x, d(y)]_\alpha\beta d(z) + 2z\gamma x\beta z\alpha y = 0,$$

for all  $x, y, z \in U$  and  $\alpha, \beta, \gamma \in \Gamma$ . If we compare (25) and (26), we have that

$$(27) \quad d(z)\gamma x\beta[z, d(y)]_\alpha - [z, d(y)]_\alpha\gamma x\beta d(z) = 0,$$

for all  $x, y, z \in U$  and  $\alpha, \beta, \gamma \in \Gamma$ . Replacing  $x$  by  $2x\sigma[z, d(y)]_\alpha$  for  $y \in [U, U]_\Gamma$  and  $\sigma \in \Gamma$  in the equation (27) we get

$$(28) \quad d(z)\gamma x\sigma[z, d(y)]_\alpha\beta[z, d(y)]_\alpha - [z, d(y)]_\alpha\gamma x\sigma[z, d(y)]_\alpha\beta d(z) = 0,$$

since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Taking  $\sigma$  for  $\beta$  in (27) we have

$$(29) \quad d(z)\gamma x\sigma[z, d(y)]_\alpha = [z, d(y)]_\alpha\gamma x\sigma d(z).$$

If we use the equation (29) in the equation (28) we get

$$[z, d(y)]_\alpha\gamma x\sigma d(z)\beta[z, d(y)]_\alpha = [z, d(y)]_\alpha\gamma x\sigma[z, d(y)]_\alpha\beta d(z)$$

and so

$$(30) \quad [z, d(y)]_\alpha\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\beta = 0,$$

for all  $x, z \in U$ ,  $y \in [U, U]_\Gamma$  and  $\alpha, \beta, \gamma, \sigma \in \Gamma$ . Taking  $\beta = \gamma$  in (30), we get

$$(31) \quad [z, d(y)]_\alpha\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0,$$

for all  $x, z \in U$ ,  $y \in [U, U]_\Gamma$  and  $\alpha, \gamma, \sigma \in \Gamma$ . Multiplying the equation (31) on the left by  $d(z)\gamma$  for  $z \in [U, U]_\Gamma$ , we have

$$(32) \quad d(z)\gamma[z, d(y)]_\alpha\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0.$$

Taking  $2d(z)\gamma x$  for  $x$  in (31) we obtain that

$$(33) \quad [z, d(y)]_\alpha\gamma d(z)\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0,$$

for all  $x \in U$ ,  $y, z \in [U, U]_\Gamma$  and  $\alpha, \gamma, \sigma \in \Gamma$  since  $M$  is a  $\Gamma$ -ring of characteristic not 2. Subtracting (33) from (32) we see that

$$[d(z), [z, d(y)]_\alpha]_\gamma\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0,$$

for all  $x \in U$ ,  $y, z \in [U, U]_\Gamma$  and  $\alpha, \gamma, \sigma \in \Gamma$ . Therefore, by Lemma 3.4 we have that

$$[d(z), [z, d(y)]_\alpha]_\gamma = 0,$$

for all  $y, z \in [U, U]_\Gamma$  and  $\alpha, \gamma \in \Gamma$ . Then, the proof is completed by using the similar steps in the equation (22) in Theorem 3.3.  $\square$

**Corollary 3.4.** *Let  $d$  be a derivation of a prime  $\Gamma$ -ring  $M$  of characteristic not 2. If  $d(x) \circ_\alpha d(y) = -(x \circ_\alpha y)$  for all  $x, y \in M$ ,  $\alpha \in \Gamma$  and  $Z(M) \neq (0)$ , then  $M$  is commutative.*

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